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Yuji Ishiguro and Roberto D M Garcia

CENTRO DE ENGENHARIA NUCLEAR
CEN AFR 060

INSTITUTO DE ENERGIA ATÔMICA
SÃO PAULO – BRASIL

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Two Media Problems in Two Group Neutron Transport Theory

Yuji Ishiguro and Roberto D. M. Garcia
Instituto de Energia Atomica, Cidade Universitaria, São Paulo, Brazil

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ABSTRACT

Three neutron transport problems involving two different media are solved in two group theory for isotropic scattering, based on the singular eigenfunction expansion solution of the transport equation. This work has two purposes. First, it is shown that two media problems in two group theory can be reduced to regular computational forms using the half range orthogonality theorem; second, in support of benchmark activities, three model problems are defined and their solutions are reported based on an exact theory.

I. INTRODUCTION

The two group neutron transport equation for isotropic scattering in plane geometry has been studied by many researchers in the singular eigenfunction expansion method.[1] The first work was reported soon after the introduction of the method by Zelazny and Kuszell,[2] but their completeness arguments were not quite conclusive. Some years later, Siewert and Shieh,[3] following the work of Siewert and Zweifel[4] on a special case of the multigroup model, rigorously proved the full range completeness and orthogonality theorems and analyzed the discrete spectrum. Some attempts were made to solve half space[5] and slab[6-8] problems, but it was not until the half range completeness and orthogonality theorems were established[9-12] that these problems were solved in a concise manner. Half space problems[10] are solved in terms of an H matrix that can be obtained numerically by a rapidly converging iterative scheme, and slab problems[13] can be converted to systems of regular integral equations for the expansion coefficients, which can then be solved by numerical iterations.

Problems involving two media, however, have remained unsolved in two group theory, although in the one group model some problems have been solved using the two media orthogonality relations[14] and by other methods.[15,16] The difficulty is that the use of the full range and half range orthogonality relations does not remove all singularities that are inherent in the Case method, and that the numerical solution of the resulting singular integral equations involves numerical differentiations. Furthermore, two media orthogonality relations have not been found in two group theory. Jauho and Rajamäki[17] studied two media problems in multigroup theory, but they did not report any numerical results, and it appears rather difficult to obtain numerical results based on their analysis. The first numerical results for two media problems were reported by Ishiguro and Maiorino,[18] using a method based on the half range orthogonality relations and invariance principles. Their method, however, is applicable only to two half space problems. Thus, a general systematic method to solve various two- or multislab problems has been lacking, and many model problems in transport theory have remained unsolved.

In a recent paper,[19] Ishiguro proposed a method of this kind and reported some numerical solutions in one group theory. In the present Note, we show that two media problems in two group neutron transport theory for isotropic scattering can be converted in a similar manner to a set of regular integral equations for the coefficients of the Case expansions and solved numerically by a standard iterative method. We report numerical results for three model problems based on exact theory: a two region slab with an incident flux, criticality for reflected slab reactors, and the cell problem. We begin by summarizing the method of regularization and the basic theory.

I.A. *The Method of Regularization*

The method to derive a set of regular integral equations for the expansion coefficients from the set of singular integral equations that results from boundary and interface conditions can be summarized in the following steps[19]:

1. At an interface, separate the continuity condition into two equations, one for $\mu \in (0,1)$ and the other for $\mu \in (-1,0)$.

2a. To the $\mu \in (0,1)$ equation, apply the half range orthogonality relations for the right medium.

2b. In the $\mu \in (-1,0)$ equation, change μ to $-\mu$ and then apply the orthogonality relations for the left medium.

3a. If any singularity remains in step 2a, consider the interface (or boundary) condition for $\mu > 0$ at the left boundary of the left medium and generate the same singularity, subtract the result from the equation in step 2a, and remove the singularity.

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3b For step 2b consider the right interface of the right medium and generate the same singularity from the $\mu < 0$ equation

4 If singularities remain in step 3 repeat the process generating the same singularities at different interfaces

Although the equation for a discrete coefficient is always found to be regular we apply to this equation the same operations as those applied to the equation for the corresponding continuum coefficient since the convergence of iterations is sometimes faster and the discrete and continuum coefficients are obtained in the same form We note that for a symmetric geometry the right and left interfaces are equivalent

I B Solution of the Transport Equation

The two group neutron transport equation for isotropic scattering can be written as

$$\mu \frac{\partial}{\partial x} I(x, \mu) + \Sigma I(x, \mu) = Q \int_{-1}^{1} I(x, \mu')\, d\mu' \qquad (1)$$

where the space variable x is measured in units of the mean free path for group 2 neutrons As in previous works[10,12,14] we assume that the scattering matrix Q is neither diagonal nor triangular and that det $Q \neq 0$ and introduce a matrix P defined as

$$\mathbf{P} = \begin{bmatrix} (q_{12}/q_{21})^{1/2} & 0 \\ 0 & 1 \end{bmatrix} \qquad (2)$$

where q_{ij} are the elements of Q Then the solution of Eq (1) is given by

$$I(x, \mu) = P^{-1} \Psi(x, \mu) \qquad (3)$$

where $\Psi(x, \mu)$ is the solution of

$$\mu \frac{\partial}{\partial x} \Psi(x, \mu) + \Sigma \Psi(x, \mu) = C \int_{-1}^{1} \Psi(x, \mu')\, d\mu' \qquad (4)$$

with the symmetrized scattering matrix given by $C = PQP^{-1}$ and where the elements of Σ are $\Sigma_{11} = \sigma$ $\Sigma_{12} = \Sigma_{21} = 0$ and $\Sigma_{22} = 1$

The general solution of Eq (4) can be written[9,10] as

$$\begin{aligned} \Psi(x, \mu) = \sum_{j=1}^{\kappa} & [A(\nu_j)\Phi(\nu_j, \mu) \exp(-x/\nu_j) \\ & + A(-\nu_j)\Phi(-\nu_j, \mu) \exp(x/\nu_j)] \\ & + \int_{①} [A_1^{(1)}(\nu)\Phi_1^{(1)}(\nu, \mu) \exp(-x/\nu) \\ & + A_2^{(1)}(\nu)\Phi_2^{(1)}(\nu, \mu) \exp(-x/\nu)]\, d\nu \\ & + \int_{②} A^{(2)}(\nu)\Phi^{(2)}(\nu, \mu) \exp(-x/\nu)\, d\nu \end{aligned} \qquad (5)$$

where the A s are expansion coefficients to be determined by the boundary condition once a specific problem is considered and discrete eigenvalues $\pm\nu_j$ are the zeros of det $\Lambda(z)$ with

$$\Lambda(z) = I - z \int_{-1}^{1} K(z, \mu)\, d\mu\, C \qquad (6)$$

where κ (either 1 or 2)[4] is the number of pairs of the discrete eigenvalues and the eigenfunctions can be written as

$$\Phi(\pm\nu_j, \mu) = \nu_j K(\nu_j, \pm\mu) C U(\nu_j) \qquad (7a)$$

$$\Phi_\alpha^{(1)}(\nu, \mu) = [\nu K(\nu, \mu) C + \delta(\nu, \mu)\lambda(\nu)] U_\alpha^{(1)}(\nu) \qquad \alpha = 1, 2$$

$$\nu \in \text{Region } ① = (-1/\sigma, 1/\sigma) \qquad (7b)$$

and

$$\Phi^{(2)}(\nu, \mu) = [\nu K(\nu, \mu) C + \delta(\nu, \mu)\lambda(\nu)] U^{(2)}(\nu)$$

$$\nu \in \text{Region } ② = (-1, -1/\sigma) \cup (1/\sigma, 1) \qquad (7c)$$

Here

$$K(\xi, \mu) = \begin{bmatrix} \dfrac{\mu}{\sigma\xi - \mu} & 0 \\ 0 & \dfrac{\mu}{\xi - \mu} \end{bmatrix} \qquad (8a)$$

$$\delta(\nu, \mu) = \begin{bmatrix} \delta(\sigma\nu - \mu) & 0 \\ 0 & \delta(\nu - \mu) \end{bmatrix} \qquad (8b)$$

$$U_1^{(1)}(\nu) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad (9a)$$

$$U_2^{(1)}(\nu) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad (9b)$$

$$U^{(2)}(\nu) = \begin{bmatrix} \lambda_{22}(\nu) \\ -\lambda_{21}(\nu) \end{bmatrix} \qquad (9c)$$

$$U(\nu_j) = \begin{bmatrix} \Lambda_{22}(\nu_j) \\ -\Lambda_{21}(\nu_j) \end{bmatrix} \qquad (9d)$$

and

$$\lambda(\nu) = I - \nu \int_{-1}^{1} K(\nu, \mu)\, d\mu\, C \qquad (10)$$

with I being the 2 × 2 identity matrix

The full range and half range completeness and orthogonality theorems regarding the solution given by Eq (5) have been established[9,10,13]

Although the solution has been used in previous works[10,12,14] in the form of Eq (5) we write the general solution in a more compact form as

$$\begin{aligned} \Psi(x, \mu) = \sum_{j=1}^{\kappa} & [A(\nu_j)\Phi(\nu_j, \mu) \exp(-x/\nu_j) \\ & + A(-\nu_j)\Phi(-\nu_j, \mu) \exp(x/\nu_j)] \\ & + \int_0^1 \Phi(\nu, \mu) A(\nu) \exp(-x/\nu)\, d\nu \\ & + \int_0^1 \Phi(-\nu, \mu) A(-\nu) \exp(x/\nu)\, d\nu \end{aligned} \qquad (11)$$

where the discrete eigenfunctions are the same as in Eq (7a) the continuum eigenfunction is a 2 × 2 matrix defined as

$$\Phi(\nu, \mu) = \nu K(\nu, \mu) C + \delta(\nu, \mu)\lambda(\nu) \qquad \nu \in (-1, 1) \qquad (12)$$

and $A(\pm\nu)$ are two-vector expansion coefficients We note that the expansion given in Eq (11) is not the general solution of Eq (4) if $A(\pm\nu)$ are arbitrary for $\nu \in (1/\sigma, 1)$ However as later equations show $A(\pm\nu)$ are always found in our formalism to be proportional to $U^{(2)}(\nu)$ for $\nu \in (1/\sigma, 1)$ and thus considering Eqs (5) (7) and (9) we can write Eq (5) in the more compact form of Eq (11) We always separate positive and negative eigenvalues as in Eq (11) and use the symbols ν ξ and η to denote positive eigenvalues

I C The H Matrix

The H matrix introduced in Ref 10 plays a principal role in the half range orthogonality theorem and has been discussed in detail in Ref 11 We list some of the equations it satisfies for use in our problems

The H matrix satisfies the integral equations

$$\tilde{\mathbf{H}}(z)\Lambda(z) = \mathbf{I} + z\int \tilde{\mathbf{H}}(\mu)\boldsymbol{\theta}(\mu)\frac{d\mu}{\mu - z}\mathbf{C} \qquad z \notin (0, 1) \tag{13a}$$

and

$$\nu \int \tilde{\mathbf{H}}(\mu)\boldsymbol{\theta}(\mu)\frac{d\mu}{\nu - \mu}\mathbf{C}\mathbf{U}(\nu) = \mathbf{U}(\nu) \tag{13b}$$

where

$$\boldsymbol{\theta}(\mu) = \begin{bmatrix} \theta(\mu) & 0 \\ 0 & 1 \end{bmatrix}$$

$\theta(\mu) = 1$ for $\mu \in (0, 1/\sigma)$ and $\theta(\mu) = 0$ otherwise (14)

To calculate the $\mathbf{H}$ matrix numerically we can use the equation

$$\mathbf{H}(z) = \mathbf{I} + z\mathbf{H}(z)\mathbf{C}\int_0 \tilde{\mathbf{H}}(\mu)\boldsymbol{\theta}(\mu)\frac{d\mu}{\mu + z} \qquad z \notin (-1, 0) \tag{15}$$

The dispersion matrix $\Lambda(z)$ can be factored in terms of the $\mathbf{H}$ matrix as

$$\mathbf{H}(-z)\mathbf{C}\tilde{\mathbf{H}}(z)\Lambda(z) = \mathbf{C} \qquad z \notin (-1, 1) \tag{16}$$

If we let $z \to \nu \pm i0$ in Eq. (13a) we can find

$$\tilde{\mathbf{H}}(\nu)\lambda(\nu) = \mathbf{I} + \nu\int_0^1 \tilde{\mathbf{H}}(\mu)\boldsymbol{\theta}(\mu)\frac{P}{\mu - \nu}d\mu\mathbf{C} \qquad \nu \in (0, 1) \tag{17}$$

Since the existence of a unique solution of these equations has been established we use them freely in our problem for example we have from Eq. (15)

$$z\int_0 \tilde{\mathbf{H}}(\mu)\boldsymbol{\theta}(\mu)\frac{d\mu}{\mu + z}\mathbf{k} = \mathbf{C}^{-1}\mathbf{k} - \mathbf{C}^{-1}\mathbf{H}^{-1}(z)\mathbf{k} \qquad z \notin (-1, 0) \tag{18a}$$

and from Eq. (17)

$$\nu\int_0 \tilde{\mathbf{H}}(\mu)\boldsymbol{\theta}(\mu)\frac{P}{\nu - \mu}d\mu\mathbf{k} = \mathbf{C}^{-1}\mathbf{k} - \tilde{\mathbf{H}}(\nu)\lambda(\nu)\mathbf{C}^{-1}\mathbf{k} \qquad \nu \in (0, 1) \tag{18b}$$

for an arbitrary 2×2 matrix $\mathbf{k}$. We call these equations collectively the $\mathbf{H}$ equations.

I D Half Range Orthogonality and Related Integrals

Half range orthogonality relations of the eigenfunctions are given in Ref. 10. However since we use a different form to write the solution we redefine the adjoint functions.

The discrete adjoint vector is the same as in Ref. 10

$$\boldsymbol{\phi}(\nu_j, \mu) = \nu_j\mathbf{K}(\nu_j, \mu)\mathbf{h}(\mu)\mathbf{H}^{-1}(\nu_j)\mathbf{C}\mathbf{U}(\nu_j) \qquad \nu_j > 1 \text{ or } \nu_j = \varepsilon(\sigma_j) \tag{19a}$$

We define the continuum adjoint matrix as

$$\boldsymbol{\phi}(\nu, \mu) = [\nu\mathbf{K}(\nu, \mu)\mathbf{h}(\mu)\mathbf{H}^{-1}(\nu)\mathbf{C} + \delta(\nu - \mu)\lambda(\nu)]\mathbf{W}(\nu) \qquad \nu \in (0, 1) \tag{19b}$$

where the symmetric matrix

$$\mathbf{W}(\nu) = \begin{bmatrix} N_{22}(\nu) & N_{21}(\nu) \\ N_{12}(\nu) & N_{11}(\nu) \end{bmatrix}\theta(\nu) + \mathbf{U}^{(2)}(\nu)\tilde{\mathbf{U}}^{(2)}(\nu)[1 - \theta(\nu)] \tag{20}$$

is the same matrix as was used in Ref. 13 and

$$\mathbf{h}(\mu) = \begin{bmatrix} H_{11}(\mu/\sigma) & H_{12}(\mu/\sigma) \\ H_{21}(\mu) & H_{22}(\mu) \end{bmatrix} \qquad \mu \in (0, 1) \tag{21}$$

with H_{ij} being the elements of the $\mathbf{H}$ matrix.

With these adjoint functions the orthogonality relations can be written as

$$\int \tilde{\boldsymbol{\phi}}(\nu_i, \mu)\boldsymbol{\Phi}(\nu_j, \mu)\mu d\mu = N(\nu_i)\delta_{ij} \tag{22a}$$

$$\int_0^1 \tilde{\boldsymbol{\phi}}(\nu, \mu)\boldsymbol{\Phi}(\nu_i, \mu)\mu d\mu = \mathbf{0} \tag{22b}$$

$$\int^1 \tilde{\boldsymbol{\phi}}(\nu_i, \mu)\boldsymbol{\Phi}(\nu, \mu)\mu d\mu = \mathbf{0} \tag{22c}$$

and

$$\int_0 \tilde{\boldsymbol{\phi}}(\nu, \mu)\boldsymbol{\Phi}(\nu', \mu)\boldsymbol{A}(\nu')\mu d\mu = \mathbf{N}(\nu)\boldsymbol{A}(\nu)\delta(\nu - \nu') \tag{22d}$$

where $\boldsymbol{A}(\nu)$ in the last formula is an arbitrary two vector and the N functions in Eqs. (20), (22a) and (22d) are given explicitly in Refs. 10 and 13.

Since we need various half range integrals of the product of eigenfunction and adjoint function we summarize some of these formulas here. To simplify the notation we let

$$\tilde{X}(\nu_i) = \tilde{\mathbf{U}}(\nu_i)\mathbf{C}\tilde{\mathbf{H}}^{-1}(\nu_i)\mathbf{C}^{-1} \tag{23a}$$

and

$$\tilde{\mathbf{X}}(\nu) = \tilde{\mathbf{W}}(\nu)\mathbf{C}\tilde{\mathbf{H}}^{-1}(\nu)\mathbf{C}^{-1} \tag{23b}$$

When the eigenfunction and adjoint belong to the same medium we can evaluate the following integrals using the $\mathbf{H}$ equations to obtain

$$\int \tilde{\boldsymbol{\phi}}(\nu_i, \mu)\boldsymbol{\Phi}(-\nu_j, \mu)\mu d\mu = \frac{\nu_i\nu_j}{\nu_i + \nu_j}\tilde{X}(\nu_i)\mathbf{H}^{-1}(\nu_j)\mathbf{C}\mathbf{U}(\nu_j) \tag{24a}$$

$$\int \tilde{\boldsymbol{\phi}}(\nu_i, \mu)\boldsymbol{\Phi}(-\nu, \mu)\mu d\mu = \frac{\nu_i\nu}{\nu_i + \nu}\tilde{X}(\nu_i)\mathbf{H}^{-1}(\nu)\mathbf{C} \tag{24b}$$

$$\int \tilde{\boldsymbol{\phi}}(\nu, \mu)\boldsymbol{\Phi}(-\nu_i, \mu)\mu d\mu = \frac{\nu\nu_i}{\nu + \nu_i}\tilde{\mathbf{X}}(\nu)\mathbf{H}^{-1}(\nu_i)\mathbf{C}\mathbf{U}(\nu_i) \tag{24c}$$

and

$$\int_0^1 \tilde{\boldsymbol{\phi}}(\nu, \mu)\boldsymbol{\Phi}(-\nu', \mu)\mu d\mu = \frac{\nu\nu'}{\nu + \nu'}\tilde{\mathbf{X}}(\nu)\mathbf{H}^{-1}(\nu')\mathbf{C} \tag{24d}$$

If the eigenfunction and adjoint belong to different media the integral of their product is more involved. All integrals can be performed however if we decompose the $\mathbf{K}$ matrix as

$$\mathbf{K}(\xi, \mu) = \frac{P}{\sigma\xi - \mu}\mathbf{k}_1 + \frac{P}{\xi - \mu}\mathbf{k}_2 \tag{25a}$$

with

$$\mathbf{k}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{25b}$$

and

$$\mathbf{k}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{25c}$$

and use the $\mathbf{H}$ equations e.g. Eqs. (18). Since these formulas are rather lengthy and since the later equations for the three problems show most of them clearly we report here only one, the simplest

$$\int \mu\tilde{\boldsymbol{\phi}}_1(\nu_i, \mu)\mathbf{G}\boldsymbol{\Phi}_2(-\eta_j, \mu)\, d\mu = \nu_i\tilde{X}_1(\nu_i)\left[\frac{\sigma_2\eta_j}{\sigma_2\eta_j + \sigma_1\nu_i}\mathbf{H}_2^{-1}(\sigma_2\eta_j/\sigma_1)\mathbf{k}_1 + \frac{\eta_j}{\eta_j + \nu_i}\mathbf{H}_2^{-1}(\eta_j)\mathbf{k}_2\right] \times \mathbf{G}\mathbf{C}_2\mathbf{U}_2(\eta_j) \tag{26}$$

where $\mathbf{G}$ is a diagonal 2×2 matrix and the subscripts are used to refer to the media.

We note that among the various integrals involving eigenfunction and adjoint only the following two are singular after integration over μ

$$\int_0^1 \mu \tilde{\theta}(\nu, \mu) \int_0^1 \Phi_j(\eta, \mu) A_j(\eta)\, d\eta\, d\mu \qquad \nu, \eta \in (0, 1) \qquad i \neq j \tag{27}$$

and

$$\int_0^1 \mu \tilde{\theta}(\nu, \mu) E(\nu) \int^1 \Phi_i(\nu', \mu) A_i(\nu')\, d\nu'\, d\mu \qquad \nu, \nu' \in (0, 1) \tag{28}$$

where $E(\nu)$ is a 2×2 matrix. Here we notice a difference between one group and two group theories in that in one group theory the integral corresponding to Eq. (28) is regular since it reduces to one corresponding to Eq. (22d). Finally, the following integral is of interest

$$\int_0^1 \tilde{\theta}(\xi, \mu) \mu\, d\mu = \xi \tilde{X}(\xi)(I - C\tilde{H}_0) Z \tag{29}$$

where H_0 is a moment of the H matrix

$$H_0 = \int_0^1 \theta(\mu) H(\mu)\, d\mu \tag{30}$$

II. THE TWO SLAB PROBLEM

We consider a slab of thickness a_1 of medium 1 $(0 \le x \le a_1)$ adjacent to another of thickness a_2 of medium 2 $(a_1 \le x \le \gamma,\ \gamma = a_1 + a_2)$ irradiated on the $x = 0$ surface by a flux of neutrons $f(\mu)$, $\mu \in (0, 1)$.

We write the solutions of Eq. (4) as

$$\begin{aligned} \Phi_1(x, \mu) = \sum_{i=1}^{1} \{ & A_1(\nu_i) \Phi_1(\nu_i, \mu) \exp(-x/\nu_i) \\ & + A_1(-\nu_i) \Phi_1(-\nu_i, \mu) \exp[(a_1 - x)/\nu_i] \} \\ & + \int_0^1 \{ \Phi_1(\nu, \mu) A_1(\nu) \exp(-x/\nu) \\ & + \Phi_1(-\nu, \mu) A_1(-\nu) \exp[(a_1 - x)/\nu] \}\, d\nu \qquad 0 \le x \le a_1 \end{aligned} \tag{31}$$

and

$$\begin{aligned} \Phi_2(x, \mu) = \sum_{i=1}^{2} \{ & A_2(\eta_i) \Phi_2(\eta_i, \mu) \exp[-(x - a_1)/\eta_i] \\ & + A_2(-\eta_i) \Phi_2(-\eta_i, \mu) \exp[(\gamma - x)/\eta_i] \} \\ & + \int_0^1 \{ \Phi_2(\eta, \mu) A_2(\eta) \exp[-(x - a_1)/\eta] \\ & + \Phi_2(-\eta, \mu) A_2(-\eta) \exp[(\gamma - x)/\eta] \}\, d\eta \qquad a_1 \le x \le \gamma \end{aligned} \tag{32}$$

subject to the conditions

$$\Phi_1(0, \mu) = P_1 f(\mu) \qquad \mu \in (0, 1) \tag{33a}$$

$$\Phi_2(\gamma, -\mu) = 0 \qquad \mu \in (0, 1) \tag{33b}$$

and

$$\Phi_1(a_1, \mu) = G\Phi_2(a_1, \mu) \qquad \mu \in (-1, 1) \tag{33c}$$

We assume, considering the data sets for our calculations, that the groups are similarly ordered for both media and thus the matrix G is diagonal and given by $G = P_1 P_2^{-1}$.

The conditions at outer boundaries, Eqs. (33a) and (33b), result in the equations

$$\sum_{i=1}^{1} A_1(\nu_i) \Phi_1(\nu_i, \mu) + \int_0^1 \Phi_1(\nu, \mu) A_1(\nu)\, d\nu = P_1 f(\mu) - \sum_{i=1}^{1} A_1(-\nu_i) \Phi_1(-\nu_i, \mu) E_1(\nu_i) - \int_0^1 \Phi_1(-\nu, \mu) A_1(-\nu) E_1(\nu)\, d\nu \qquad \mu \in (0, 1) \tag{34}$$

and

$$\begin{aligned} & \sum_{i=1}^{2} A_2(-\eta_i) \Phi_2(\eta_i, \mu) + \int_0^1 \Phi_2(\eta, \mu) A_2(-\eta)\, d\eta \\ & = -\sum_{i=1}^{2} A_2(\eta_i) \Phi_2(-\eta_i, \mu) E_2(\eta_i) - \int_0^1 \Phi_2(-\eta, \mu) A_2(\eta) E_2(\eta)\, d\eta \\ & \qquad \mu \in (0, 1) \end{aligned} \tag{35}$$

and we write the interface condition, Eq. (33c), in two equations

$$\begin{aligned} & \sum_{i=1}^{1} A_1(-\nu_i) \Phi_1(\nu_i, \mu) + \int_0^1 \Phi_1(\nu, \mu) A_1(-\nu)\, d\nu \\ & = -\sum_{i=1}^{1} A_1(\nu_i) \Phi_1(-\nu_i, \mu) E_1(\nu_i) - \int_0^1 \Phi_1(-\nu, \mu) A_1(\nu) E_1(\nu)\, d\nu \\ & + \sum_{i=1}^{2} G [A_2(\eta_i) \Phi_2(-\eta_i, \mu) + A_2(-\eta_i) \Phi_2(\eta_i, \mu) E_2(\eta_i)] \\ & + \int_0^1 G [\Phi_2(-\eta, \mu) A_2(\eta) + \Phi_2(\eta, \mu) A_2(-\eta) E_2(\eta)]\, d\eta \ , \\ & \qquad \mu \in (0, 1) \end{aligned} \tag{36}$$

and

$$\begin{aligned} & \sum_{i=1}^{2} A_2(\eta_i) \Phi_2(\eta_i, \mu) + \int_0^1 \Phi_2(\eta, \mu) A_2(\eta)\, d\eta \\ & = \sum_{i=1}^{1} G^{-1} [A_1(\nu_i) \Phi_1(\nu_i, \mu) E_1(\nu_i) + A_1(-\nu_i) \Phi_1(-\nu_i, \mu)] \\ & + \int_0^1 G^{-1} [\Phi_1(\nu, \mu) A_1(\nu) E_1(\nu) + \Phi_1(-\nu, \mu) A_1(-\nu)]\, d\nu \\ & - \sum_{i=1}^{2} A_2(-\eta_i) \Phi_2(-\eta_i, \mu) E_2(\eta_i) \\ & - \int_0^1 \Phi_2(-\eta, \mu) A_2(-\eta) E_2(\eta)\, d\eta \qquad \mu \in (0, 1) \end{aligned} \tag{37}$$

where $E_i(\xi) = \exp(-a_i/\xi)$.

Our aim is to derive a set of regular integral equations for the expansion coefficients so that the coefficients can be found numerically by a standard iterative method. If we apply the half range orthogonality theorem to Eq. (34), i.e., multiply Eq. (34) by $\mu\tilde{\phi}_1(\xi, \mu)$, $\xi = \nu_i$ or $\nu \in (0, 1)$, we obtain

$$A_1(\nu_i) = A_1^0(\nu_i) - \nu_i N_1^{-1}(\nu_i) \tilde{X}_1(\nu_i) Y_1(\nu_i) \tag{38a}$$

and

$$A_1(\nu) = A_1^0(\nu) - \nu N_1^{-1}(\nu) \tilde{X}_1(\nu) Y_1(\nu) \ , \tag{38b}$$

and in the same way, we obtain from Eq. (35)

$$A_2(-\eta_i) = -\eta_i N_2^{-1}(\eta_i) \tilde{X}_2(\eta_i) Y_2(\eta_i) \tag{39a}$$

and

$$A_2(-\eta) = -\eta N_2^{-1}(\eta) \tilde{X}_2(\eta) Y_2(\eta) \ , \tag{39b}$$

where

$$A_1^0(\nu_i) = N_1^{-1}(\nu_i) \int_0^1 \tilde{\phi}_1(\nu_i, \mu) P_1 f(\mu) \mu\, d\mu \ , \tag{40a}$$

$$A_1^0(\nu) = N_1^{-1}(\nu) \int_0^1 \tilde{\phi}_1(\nu, \mu) P_1 f(\mu) \mu\, d\mu \tag{40b}$$

$$\begin{aligned} Y_1(\xi) = & \sum_{i=1}^{1} \frac{\nu_i}{\nu_i + \xi} H_1^{-1}(\nu_i) C_1 \theta_1(\nu_i) A_1(-\nu_i) E_1(\nu_i) \\ & + \int_0^1 \frac{\nu}{\nu + \xi} H_1^{-1}(\nu) C_1 A_1(-\nu) E_1(\nu)\, d\nu \end{aligned} \tag{41}$$

and

$$Y_2(\xi) = \sum_i \frac{\eta_i}{\eta_i + \xi} H_2^{-1}(\eta_i) C_2 U_2(\eta_i) A_2(\eta_i) E_2(\eta_i) + \int_0^1 \frac{\eta}{\eta + \xi} H_2^{-1}(\eta) C_2 A_2(\eta) E_2(\eta) d\eta \tag{42}$$

Next we apply the orthogonality theorem for medium 1 to Eq (36) to isolate the coefficients on the left side. After integrating over μ the $A_2(\eta)$ terms remain to be principal value integrals for $\xi = \nu$. Following the method of regularization summarized in Sec 1 we multiply Eq (35) by

$$\mu \widetilde{\Phi}_1(\xi, \mu) G \begin{bmatrix} E_2(\sigma_1\xi/\sigma_2) & 0 \\ 0 & E_1(\xi) \end{bmatrix} \tag{43}$$

and integrate over $\mu \in (0, 1)$. We find on the left side the same singular integrals but with different exponential functions as those from Eq (36). Then subtracting this result from the previous equation we obtain equations with removable singularities

$$A_1(\nu_i) = \nu_i N_1^{-1}(\nu_i) \widetilde{X}_1(\nu_i) Y_3(\nu_i) \tag{44a}$$

and

$$A_1(\nu) = \nu N_1^{-1}(\nu) \widetilde{X}_1(\nu) Y_3(\nu) \tag{44b}$$

where

$$Y_3(\xi) = \sum_i \frac{\nu_i}{\nu_i + \xi} H_1^{-1}(\nu_i) C_1 U_1(\nu_i) A_1(\nu_i) E_1(\nu_i) - \int_0^1 \frac{\nu}{\nu + \xi} H_1^{-1}(\nu) C_1 A_1(\nu) E_1(\nu) d\nu$$
$$+ \sum_i \left\{ \frac{\sigma_1\eta_i}{\sigma_2\eta_i + \sigma_1\xi} H_1^{-1}(\sigma_2\eta_i/\sigma_1) k_1 [1 - E_2(\sigma_1\xi/\sigma_2) E_2(\eta_i)] + \frac{\eta_i}{\eta_i + \xi} H_1^{-1}(\eta_i) k_2 [1 - E_2(\xi) E_2(\eta_i)] \right\} G C_2 U_2(\eta_i) A_2(\eta_i)$$
$$+ \sum_i \left\{ \frac{\sigma_1\eta_i}{\sigma_2\eta_i - \sigma_1\xi} H_1^{-1}(\sigma_2\eta_i/\sigma_1) k_1 [E_2(\eta_i) - E_2(\sigma_1\xi/\sigma_2)] + \frac{\eta_i}{\eta_i - \xi} H_1^{-1}(\eta_i) k_2 [E_2(\eta_i) - E_2(\xi)] \right\} G C_2 U_2(\eta_i) A_2(-\eta_i)$$
$$+ \int_0^1 \left\{ \frac{\sigma_1\eta}{\sigma_2\eta + \sigma_1\xi} H_1^{-1}(\sigma_2\eta/\sigma_1) k_1 [1 - E_2(\sigma_1\xi/\sigma_2) E_2(\eta)] + \frac{\eta}{\eta + \xi} H_1^{-1}(\eta) k_2 [1 - E_2(\xi) E_2(\eta)] \right\} G C_2 A_2(\eta) d\eta$$
$$+ \int_0^1 C_1 \left\{ \frac{\sigma_1\eta}{\sigma_2\eta - \sigma_1\xi} \widetilde{H}_1(\sigma_2\eta/\sigma_1) C_1^{-1} \widetilde{\lambda}_1(\sigma_2\eta/\sigma_1) k_1 [E_2(\eta) - E_2(\sigma_1\xi/\sigma_2)] + \frac{\eta}{\eta - \xi} \widetilde{H}_1(\eta) C_1^{-1} \widetilde{\lambda}_1(\eta) k_2 [E_2(\eta) - E_2(\xi)] \right\} G C_2 A_2(-\eta) d\eta$$
$$+ \int_0^1 C_1 \left\{ \frac{\sigma_2\eta}{\sigma_1\xi - \sigma_2\eta} \widetilde{H}_1(\sigma_2\eta/\sigma_1) k_1 [E_2(\eta) - E_2(\sigma_1\xi/\sigma_2)] + \frac{\eta}{\xi - \eta} \widetilde{H}_1(\eta) k_2 [E_2(\eta) - E_2(\xi)] \right\} \theta_2(\eta) G \lambda_2(\eta) A_2(-\eta) d\eta \tag{45}$$

In the same way we first multiply Eq (37) by $\mu \widetilde{\Phi}_2(\xi, \mu)$, $\xi = \eta$ or $\eta \in (0, 1)$ and integrate over $\mu \in (0, 1)$; next we multiply Eq (34) by

$$\mu \widetilde{\Phi}_2(\xi, \mu) G^{-1} \begin{bmatrix} E_1(\sigma_2\xi/\sigma_1) & 0 \\ 0 & E_1(\xi) \end{bmatrix} \tag{46}$$

integrate over μ, and then subtract between the two results to obtain

$$A_2(\eta_i) = A_2^0(\eta_i) + \eta_i N_2^{-1}(\eta_i) \widetilde{X}_2(\eta_i) Y_4(\eta_i) \tag{47a}$$

and

$$A_2(\eta) = A_2^0(\eta) + \eta N_2^{-1}(\eta) \widetilde{X}_2(\eta) Y_4(\eta) \tag{47b}$$

where

$$A_2^0(\eta_i) = N_2^{-1}(\eta_i) \int_0^1 \widetilde{\Phi}_2(\eta_i, \mu) G^{-1} \begin{bmatrix} E_1(\sigma_2\eta_i/\sigma_1) & 0 \\ 0 & E_1(\eta_i) \end{bmatrix} P_1 f(\mu) \mu d\mu \tag{48a}$$

$$A_2^0(\eta) = N_2^{-1}(\eta) \int_0^1 \widetilde{\Phi}_2(\eta, \mu) G^{-1} \begin{bmatrix} E_1(\sigma_2\eta/\sigma_1) & 0 \\ 0 & E_1(\eta) \end{bmatrix} P_1 f(\mu) \mu d\mu \tag{48b}$$

and

$$Y_4(\xi) = \sum_i \frac{\eta_i}{\eta_i + \xi} H_2^{-1}(\eta_i) C_2 U_2(\eta_i) A_2(-\eta_i) E_2(\eta_i) - \int_0^1 \frac{\eta}{\eta + \xi} H_2^{-1}(\eta) C_2 A_2(-\eta) E_2(\eta) d\eta$$
$$+ \sum_i \left\{ \frac{\sigma_2\nu_i}{\sigma_1\nu_i + \sigma_2\xi} H_2^{-1}(\sigma_1\nu_i/\sigma_2) k_1 [1 - E_1(\nu_i) E_1(\sigma_2\xi/\sigma_1)] + \frac{\nu_i}{\nu_i + \xi} H_2^{-1}(\nu_i) k_2 [1 - E_1(\nu_i) E_1(\xi)] \right\} G^{-1} C_1 U_1(\nu_i) A_1(-\nu_i)$$
$$+ \sum_i \left\{ \frac{\sigma_2\nu_i}{\sigma_1\nu_i - \sigma_2\xi} H_2^{-1}(\sigma_1\nu_i/\sigma_2) k_1 [E_1(\nu_i) - E_1(\sigma_2\xi/\sigma_1)] + \frac{\nu_i}{\nu_i - \xi} H_2^{-1}(\nu_i) k_2 [E_1(\nu_i) - E_1(\xi)] \right\} G^{-1} C_1 U_1(\nu_i) A_1(\nu_i)$$
$$+ \int_0^1 \left\{ \frac{\sigma_2\nu}{\sigma_1\nu + \sigma_2\xi} H_2^{-1}(\sigma_1\nu/\sigma_2) k_1 [1 - E_1(\nu) E_1(\sigma_2\xi/\sigma_1)] + \frac{\nu}{\nu + \xi} H_2^{-1}(\nu) k_2 [1 - E_1(\nu) E_1(\xi)] \right\} G^{-1} C_1 A_1(-\nu) d\nu$$
$$+ \int_0^1 C_2 \left\{ \frac{\sigma_2\nu}{\sigma_1\nu - \sigma_2\xi} \widetilde{H}_2(\sigma_1\nu/\sigma_2) C_2^{-1} \widetilde{\lambda}_2(\sigma_1\nu/\sigma_2) k_1 [E_1(\nu) - E_1(\sigma_2\xi/\sigma_1)] + \frac{\nu}{\nu - \xi} \widetilde{H}_2(\nu) C_2^{-1} \widetilde{\lambda}_2(\nu) k_2 [E_1(\nu) - E_1(\xi)] \right\} G^{-1} C_1 A_1(\nu) d\nu$$
$$+ \int_0^1 C_2 \left\{ \frac{\sigma_1\nu}{\sigma_2\xi - \sigma_1\nu} \widetilde{H}_2(\sigma_1\nu/\sigma_2) k_1 [E_1(\nu) - E_1(\sigma_2\xi/\sigma_1)] + \frac{\nu}{\xi - \nu} \widetilde{H}_2(\nu) k_2 [E_1(\nu) - E_1(\xi)] \right\} \theta_1(\nu) G^{-1} \lambda_1(\nu) A_1(\nu) d\nu \tag{49}$$

Equations (38), (39), (44) and (47) are our final equations for the coefficients. All singularities are removed in terms of the exponential function and therefore numerical iterations can be performed in a standard manner. It is clear from these equations that, as was mentioned before, the continuum coefficients for Region ② are proportional to $U^{(1)}$.

We note that if we let $\sigma_2 \to 0$, all terms in Y_2 except the first two vanish and Eq. (44) together with Eq. (38) reduces to the case of a single slab. Similarly, in the limit $a \to 0$, Eqs. (39) and (47) reduce to the case of a single slab of medium 2.

III. THE CRITICAL PROBLEM

The critical problem for bare reactors has been solved by Kriese et al.[12] We consider here the critical problem for reflected slab reactors, a typical textbook problem in diffusion theory. The core of multiplying medium 1 extends from $-a$ to $+a$, surrounded by infinite reflectors of nonmultiplying medium 2. We assume that both media are specified and thus our aim is to determine the value of a such that nontrivial solutions exist.

We write the solutions of Eq. (4) as

$$\Phi_1(x,\mu) = \sum_{i=1}^{\kappa_1} A_1(\nu_i)\Phi_1(\nu_i,\mu)\exp[-(x+a)/\nu_i] + \sum_{i=1}^{\kappa_1} A_1(\nu_i)\Phi_1(-\nu_i,\mu)\exp[(a-x)/\nu_i] + \int_0^1 \Phi_1(\nu,\mu)A_1(\nu)\exp[-(x+a)/\nu]\,d\nu + \int_0^1 \Phi_1(-\nu,\mu)A_1(\nu)\exp[(a-x)/\nu]\,d\nu \tag{50}$$

and

$$\Phi_2(x,\mu) = \sum_{i=1}^{\kappa_2} A_2(\eta_i)\Phi_2(\eta_i,\mu)\exp[-(x-a)/\eta_i] + \int_0^1 \Phi_2(\eta,\mu)A_2(\eta)\exp[-(x-a)/\eta]\,d\eta \tag{51}$$

The symmetry condition and the condition for $|x| \to \infty$ are already incorporated in the solutions, and we consider hereafter only $x > 0$. The remaining continuity condition at $x = a$ can be written in two equations for $\mu \in (0,1)$:

$$\sum_{i=1}^{\kappa_1} A_1(\nu_i)\Phi_1(\nu_i,\mu) + \int_0^1 \Phi_1(\nu,\mu)A_1(\nu)\,d\nu = -\sum_{i=1}^{\kappa_1} A_1(\nu_i)\Phi_1(-\nu_i,\mu)E(\nu_i) - \int_0^1 \Phi_1(-\nu,\mu)A_1(\nu)E(\nu)\,d\nu + \sum_{i=1}^{\kappa_2} A_2(\eta_i)\mathbf{G}\Phi_2(-\eta_i,\mu) + \int_0^1 \mathbf{G}\Phi_2(-\eta,\mu)A_2(\eta)\,d\eta \tag{52}$$

and

$$\sum_{i=1}^{\kappa_2} A_2(\eta_i)\Phi_2(\eta_i,\mu) + \int_0^1 \Phi_2(\eta,\mu)A_2(\eta)\,d\eta = \sum_{i=1}^{\kappa_1} A_1(\nu_i)\mathbf{G}^{-1}\Phi_1(\nu_i,\mu)E(\nu_i) + \sum_{i=1}^{\kappa_1} A_1(\nu_i)\mathbf{G}^{-1}\Phi_1(-\nu_i,\mu) + \int_0^1 \mathbf{G}^{-1}\Phi_1(\nu,\mu)A_1(\nu)E(\nu)\,d\nu + \int_0^1 \mathbf{G}^{-1}\Phi_1(-\nu,\mu)A_1(\nu)\,d\nu \tag{53}$$

where $E(\xi) = \exp(-2a/\xi)$. For the moment, we assume that a is a given constant and multiply Eq. (52) by $\mu\tilde{\Phi}_1(\xi,\mu)$, $\xi = \nu_i$ or $\nu \in (0,1)$, and integrate over $\mu \in (0,1)$ to obtain equations for the coefficients:

$$A_1(\nu_i)\left[1 + \frac{1}{2}\nu_i N_1^{-1}(\nu_i)\tilde{\mathbf{X}}_1(\nu_i)\mathbf{H}_1^{-1}(\nu_i)\mathbf{C}_1 U_1(\nu_i)E(\nu_i)\right] = \nu_i N_1^{-1}(\nu_i)\tilde{\mathbf{X}}_1(\nu_i)\left[\mathbf{Y}_1(\nu_i) - (\kappa_1 - 1)\frac{\nu_2}{\nu_1+\nu_2} \times \mathbf{H}_1^{-1}(\nu_2)\mathbf{C}_1 U_1(\nu_2)E(\nu_2)A_1(\nu_2)\right] \tag{54}$$

$$A_1(\nu) = \nu N_1^{-1}(\nu)\tilde{\mathbf{X}}_1(\nu)\left[\mathbf{Y}_1(\nu) - \sum_{i=1}^{\kappa_1}\frac{\nu_i}{\nu_i+\nu} \times \mathbf{H}_1^{-1}(\nu_i)\mathbf{C}_1 U_1(\nu_i)E(\nu_i)A_1(\nu_i)\right] \tag{55a}$$

and, if $\kappa_1 = 2$,

$$A_1(\nu_2)\left[1 + \frac{1}{2}\nu_2 N_1^{-1}(\nu_2)\tilde{\mathbf{X}}_1(\nu_2)\mathbf{H}_1^{-1}(\nu_2)\mathbf{C}_1 U_1(\nu_2)E(\nu_2)\right] = \nu_2 N_1^{-1}(\nu_2)\tilde{\mathbf{X}}_1(\nu_2)\left[\mathbf{Y}_1(\nu_2) - \frac{\nu_1}{\nu_1+\nu_2} \times \mathbf{H}_1^{-1}(\nu_1)\mathbf{C}_1 U_1(\nu_1)E(\nu_1)A_1(\nu_1)\right] \tag{55b}$$

where

$$\mathbf{Y}_1(\xi) = \sum_{i=1}^{\kappa_2}\left[\frac{\sigma_1\eta_i}{\sigma_2\eta_i + \sigma_1\xi}\mathbf{H}_1^{-1}(\sigma_2\eta_i/\sigma_1)\mathbf{k}_1 + \frac{\eta_i}{\eta_i+\xi}\mathbf{H}_1^{-1}(\eta_i)\mathbf{k}_2\right] \times \mathbf{G}\mathbf{C}_2 U_2(\eta_i)A_2(\eta_i) + \int_0^1\left[\frac{\sigma_1\eta}{\sigma_2\eta+\sigma_1\xi}\mathbf{H}_1^{-1}(\sigma_2\eta/\sigma_1)\mathbf{k}_1 + \frac{\eta}{\eta+\xi}\mathbf{H}_1^{-1}(\eta)\mathbf{k}_2\right] \times \mathbf{G}\mathbf{C}_2 A_2(\eta)\,d\eta - \int_0^1 \frac{\nu}{\nu+\xi}\mathbf{H}_1^{-1}(\nu)\mathbf{C}_1 A_1(\nu)E(\nu)\,d\nu \tag{56}$$

Similarly, we multiply Eq. (53) by $\mu\tilde{\Phi}_2(\xi,\mu)$, $\xi = \eta_i$ or $\eta \in (0,1)$, and integrate over μ to isolate the coefficients in the left expansion. The $A_1(\nu)$ term on the right side remains singular. Next, we multiply Eq. (52) by

$$\mu\tilde{\Phi}_2(\xi,\mu)\mathbf{G}^{-1}\begin{bmatrix} E(\sigma_2\xi/\sigma_1) & 0 \\ 0 & E(\xi) \end{bmatrix} \tag{57}$$

and integrate over $\mu \in (0,1)$. On the left side, we find the same singular integrals with different exponential functions as in the previous equation. All other terms are regular. Then subtracting the last equation from the previous one, we obtain equations with removable singularities:

$$A_2(\eta_i)\left[1 - \frac{1}{2}\eta_i N_2^{-1}(\eta_i)\tilde{\mathbf{X}}_2(\eta_i)\mathbf{H}_2^{-1}(\eta_i)\mathbf{E}(\eta_i)\mathbf{C}_2 U_2(\eta_i)\right] = \eta_i N_2^{-1}(\eta_i)\tilde{\mathbf{X}}_2(\eta_i)\left[\mathbf{Y}_2(\eta_i) + \sum_{j=1}^{\kappa_2}\frac{\eta_j}{\eta_i+\eta_j}(1-\delta_{ij}) \times \mathbf{H}_2^{-1}(\eta_j)\mathbf{E}(\eta_j)\mathbf{C}_2 U_2(\eta_j)A_2(\eta_j)\right] \tag{58a}$$

and

$$A_2(\eta) = \eta N_2^{-1}(\eta)\tilde{\mathbf{X}}_2(\eta) \times \left[\mathbf{Y}_2(\eta) + \sum_{i=1}^{\kappa_2}\frac{\eta_i}{\eta_i+\eta}\mathbf{H}_2^{-1}(\eta_i)\mathbf{E}(\eta)\mathbf{C}_2 U_2(\eta_i)A_2(\eta_i)\right] \tag{58b}$$

where

$$\mathbf{E}(\xi) = \begin{bmatrix} E(\sigma_2\xi/\sigma_1) & 0 \\ 0 & E(\xi) \end{bmatrix} \tag{59}$$

and where

$$Y_2(\xi) = \sum \left\{\frac{\sigma_2\nu}{\sigma_1\nu + \sigma_2\xi} H(\sigma_1\nu/\sigma_2)k_1[1 - E(\nu)E(\sigma_2\xi/\sigma_1)] + \frac{\nu}{\nu + \xi} H(\nu)k_2[1 - E(\nu)E(\xi)]\right\} G^{-1}C_1U(\nu)A(\nu)$$
$$+ \sum \left\{\frac{\sigma_2\nu}{\sigma_1\nu - \sigma_2\xi} H(\sigma_1\nu/\sigma_2)k[E(\nu) - E(\sigma_2\xi/\sigma_1)] + \frac{\nu}{\nu - \xi} H_2^{-1}(\nu)k_2[E(\nu) - E(\xi)]\right\} G^{-1}C_1U_1(\nu)A_1(\nu_i)$$
$$+ \int_0 \left\{\frac{\sigma_2\nu}{\sigma_1\nu + \sigma_2\xi} H^{-1}(\sigma_1\nu/\sigma_2)k[1 - E(\nu)E(\sigma_2\xi/\sigma_1)] + \frac{\nu}{\nu + \xi} H_2(\nu)k_2[1 - E(\nu)E(\xi)]\right\} G^{-1}C_1A_1(\nu)d\nu$$
$$+ \int_0^1 C\left\{\frac{\sigma_2\nu}{\sigma_1\nu - \sigma_2\xi} \bar{H}(\sigma_1\nu/\sigma_2)C_2^{-1}\tilde{X}_2(\sigma_1\nu/\sigma_2)k_1[E(\nu) - E(\sigma_2\xi/\sigma_1)] + \frac{\nu}{\nu - \xi} \bar{H}_2(\nu)C^{-1}\tilde{X}_2(\nu)k_2[E(\nu) - E(\xi)]\right\} G^{-1}C_1A(\nu)d\nu$$
$$+ \int^1 C_2\left\{\frac{\sigma_2\nu}{\sigma_2\xi - \sigma_1\nu} \tilde{H}_2(\sigma_1\nu/\sigma_2)k[E(\nu) - E(\sigma_2\xi/\sigma_1)] + \frac{\nu}{\xi - \nu} \tilde{H}_2(\nu)k_2[E(\nu) - E(\xi)]\right\} G^{-1}\phi(\nu)\lambda_1(\nu)A_1(\nu)d\nu$$
$$+ \int_0^1 \frac{\eta}{\eta + \xi} H_2^{-1}(\eta)E(\xi)C_2A_2(\eta)d\eta \qquad (60)$$

The condition of criticality can be incorporated as the condition of nontriviality of the solution. If we normalize the solution by taking $A_1(\nu_1) = \exp(\sigma/\nu_1)$, the critical half thickness of the core is given by

$$\sigma = \frac{\pi}{2}|\nu_1| + \frac{\nu_1}{2}\ln(N/D) \qquad (61)$$

where

$$N = \frac{1}{2}\nu_1 N_1^{-1}(\nu_1)\tilde{X}_1(\nu_1)H_1^{-1}(\nu_1)C_1U_1(\nu_1) \qquad (62a)$$

and

$$D = 1 + \nu_1 N_1^{-1}(\nu_1)\tilde{X}_1(\nu_1)\exp(-\sigma/\nu_1)$$
$$\times \left[Y_1(\nu_1) - (\kappa_1 - 1)\frac{\nu_1}{\nu_1 + \nu_1} H_1^{-1}(\nu_1)C_1U_1(\nu_1)E(\nu_1)A_1(\nu_1)\right] \qquad (62b)$$

Equations (55), (59) and (61) are our final equations to be solved by numerical iterations.

III A. *The Case of Finite Reflector*

If the thickness of the reflector is finite, the core solution, Eq. (50), is the same but that for the reflector has in addition to the expansion in Eq. (51) two more terms corresponding to negative eigenvalues. Consequently the equation for $A_1(\nu)$ must be regularized once and that for $A_2(\eta)$ in two steps. In addition we have equations for $A_2(-\eta_i)$ and $A_2(-\eta)$, which are of the same form as Eqs. (39). We do not list these equations here but simply note that the functionals $Y_1(\xi)$ and $Y_2(\xi)$ have some additional terms and that numerical solutions can be obtained in the same way as for the case of infinite reflectors.

IV. THE CELL PROBLEM

We consider here an infinitely repeating array of two slabs of dissimilar media as a simplified model of flat plate fuel assemblies and analyze a unit cell consisting of a half slab of medium 1 ($-a_1 \le x \le 0$) and a half slab of medium 2 ($0 \le x \le a_2$), with the condition of symmetry with respect to the boundary surfaces. We assume uniform sources of neutrons in medium 2.

The symmetric solutions can be written as

$$I_1(x, \mu) = P_1^{-1}\Phi_1(x, \mu) \qquad -a_1 \le x \le 0 \qquad (63)$$

and

$$I(x, \mu) = P_2^{-1}[\Phi_2(x, \mu) + \Phi_{2P}(x, \mu)] \qquad 0 \le x \le a_2 \qquad (64)$$

where

$$\Phi_1(x, \mu) = \sum_1^1 A_1(\nu_i)\{\Phi_1(\nu_i, \mu)\exp[-(x + 2a_1)/\nu_i]$$
$$+ \Phi_1(-\nu_i, \mu)\exp(x/\nu_i)\}$$
$$+ \int_0^1 \{\Phi_1(\nu, \mu)\exp[-(x + 2a_1)/\nu]$$
$$+ \Phi_1(-\nu, \mu)\exp(x/\nu)\}A_1(\nu)d\nu \qquad (65)$$

$$\Phi_2(x, \mu) = \sum_1^2 A_2(\eta_i)\{\Phi_2(\eta_i, \mu)\exp(-x/\eta_i)$$
$$+ \Phi_2(-\eta_i, \mu)\exp[-(2a_2 - x)/\eta_i]\}$$
$$+ \int_0^1 \{\Phi_2(\eta, \mu)\exp(-x/\eta) + \Phi_2(-\eta, \mu)\exp[-(2a_2 - x)/\eta]\}$$
$$\times A_2(\eta)d\eta \qquad (66)$$

and

$$\Phi_{2P}(x, \mu) = (\Sigma_2 - 2C_2)^{-1}P_2S \qquad (67)$$

with S being a constant two-vector.

We write the continuity condition in two equations for $\mu \in (0, 1)$:

$$\sum_{i=1}^1 A_1(\nu_i)\Phi_1(\nu_i, \mu) + \int_0^1 \Phi_1(\nu', \mu)A_1(\nu')d\nu'$$
$$= G\Phi_{2P}(0, \mu) - \sum_{i=1}^1 A_1(\nu_i)\Phi_1(-\nu_i, \mu)E_1(\nu_i)$$
$$- \int_0^1 \Phi_1(-\nu', \mu)A_1(\nu')E_1(\nu')d\nu'$$
$$+ \sum_{i=1}^2 A_2(\eta_i)G[\Phi_2(-\eta_i, \mu) + \Phi_2(\eta_i, \mu)E_2(\eta_i)]$$
$$+ \int_0^1 G[\Phi_2(-\eta', \mu) + \Phi_2(\eta', \mu)E_2(\eta')]A_2(\eta')d\eta' \qquad (68)$$

and

$$\sum_{i=1}^2 A_2(\eta_i)\Phi_2(\eta_i, \mu) + \int_0^1 \Phi_2(\eta', \mu)A_2(\eta')d\eta'$$
$$= -\Phi_{2P}(0, \mu) + \sum_{i=1}^1 A_1(\nu_i)G^{-1}[\Phi_1(\nu_i, \mu)E_1(\nu_i) + \Phi_1(-\nu_i, \mu)]$$
$$+ \int_0^1 G^{-1}[\Phi_1(\nu', \mu)E_1(\nu') + \Phi_1(-\nu', \mu)]A_1(\nu')d\nu'$$
$$- \sum_{i=1}^2 A_2(\eta_i)\Phi_2(-\eta_i, \mu)E_2(\eta_i)$$
$$- \int_0^1 \Phi_2(-\eta', \mu)A_2(\eta')E_2(\eta')d\eta' \qquad (69)$$

where $E_i(\xi) = \exp(-2a_i/\xi)$.

In this problem, because we are actually dealing with an infinite array a straightforward application of the method of regularization requires an infinite number of steps. This is due to the fact that at each step we multiply an equation not only by the adjoint function but also by a matrix of exponential functions as in Eqs (43) (46) and (57) and that the integrals of the type that appear in Eq (28) are singular after integration over μ. In one group theory integrals of this type are regular and the regularization is accomplished after a finite number of steps even for an infinite array of multislab cells.

However the series of operations required for our problem can be summed up nicely and we can derive a regularized equation for $A_1(\nu)$ by the following steps

1 Multiply Eq (68) by $\mu\tilde{\Phi}_1(\nu,\mu)$ and integrate over μ. On the left side $A_1(\nu)$ is isolated. On the right side the $\Phi_2(\eta,\mu)$ term remains singular.

2 Multiply Eq (68) by

$$\mu\tilde{\Phi}_1(\nu,\mu)\begin{bmatrix} \dfrac{E_1(\nu)E_2(\sigma_1\nu/\sigma_2)}{1-E_1(\nu)E_2(\sigma_1\nu/\sigma_2)} & 0 \\ 0 & \dfrac{E_1(\nu)E_2(\nu)}{1-E_1(\nu)E_2(\nu)} \end{bmatrix} \tag{70}$$

and integrate over μ. The $\Phi_1(\nu',\mu)$ and $\Phi_2(\eta',\mu)$ terms remain singular.

3 Multiply Eq (69) by

$$\mu\tilde{\Phi}_1(\nu,\mu)\mathbf{G}\begin{bmatrix} \dfrac{E_2(\sigma_1\nu/\sigma_2)}{1-E_1(\nu)E_2(\sigma_1\nu/\sigma_2)} & 0 \\ 0 & \dfrac{E_2(\nu)}{1-E_1(\nu)E_2(\nu)} \end{bmatrix} \tag{71}$$

and integrate over μ. Again the $\Phi_1(\nu',\mu)$ and $\Phi_2(\eta',\mu)$ terms remain singular.

If we now add three resulting equations on each side we find an equation for $A_1(\nu)$ in which all singularities are removed in terms of exponential functions. Obviously the equation for $A_2(\nu)$ can be regularized similarly

1 Multiply Eq (69) by $\mu\tilde{\Phi}_2(\eta,\mu)$ and integrate over μ

2 Multiply Eq (69) by the following and integrate over μ

$$\mu\tilde{\Phi}_2(\eta,\mu)\begin{bmatrix} \dfrac{E_1(\sigma_2\eta/\sigma_1)E_2(\eta)}{1-E_1(\sigma_2\eta/\sigma_1)E_2(\eta)} & 0 \\ 0 & \dfrac{E_1(\eta)E_2(\eta)}{1-E_1(\eta)E_2(\eta)} \end{bmatrix} \tag{72}$$

3 Multiply Eq (68) by the following and integrate over μ

$$\mu\tilde{\Phi}_2(\eta,\mu)\mathbf{G}^{-1}\begin{bmatrix} \dfrac{E_1(\sigma_2\eta/\sigma_1)}{1-E_1(\sigma_2\eta/\sigma_1)E_2(\eta)} & 0 \\ 0 & \dfrac{E_1(\eta)}{1-E_1(\eta)E_2(\eta)} \end{bmatrix} \tag{73}$$

As in previous problems we apply these operations to the equations for the discrete coefficients also. We obtain the following equations

$$A_1(\nu_i)\left[1-\frac{1}{2}\nu_i N_1^{-1}(\nu_i)\tilde{\mathbf{X}}_1(\nu_i)\mathbf{H}_1^{-1}(\nu_i)\mathbf{J}_1(\nu_i,\nu_i)\mathbf{C}_1\mathbf{E}_1(\nu_i)\right]$$
$$= A_1^0(\nu_i)+\nu_i N_1^{-1}(\nu_i)\tilde{\mathbf{X}}_1(\nu_i)\left[\sum_{j=1}^{M_1}(1-\delta_{ij})\mathbf{Y}_1(\nu_i,\nu_j)A_1(\nu_j)\right.$$
$$+\int_0^1 \mathbf{Y}_2(\nu_i,\nu')A_1(\nu')\,d\nu'+\sum_{j=1}^{M_2}\mathbf{Y}_3(\nu_i,\eta_j)A_2(\eta_j)$$
$$\left.+\int_0^1 \mathbf{Y}_4(\nu_i,\eta')A_2(\eta')\,d\eta'\right] \tag{74a}$$

$$A_1(\nu) = A^0(\nu)+\nu N_1^{-1}(\nu)\tilde{\mathbf{X}}_1(\nu)\left[\sum_{j=1}^{M_1}\mathbf{Y}_1(\nu,\nu_j)A_1(\nu_j)\right.$$
$$+\int_0^1 \mathbf{Y}_2(\nu,\nu')A_1(\nu')\,d\nu'+\sum_{j=1}^{M_2}\mathbf{Y}_3(\nu,\eta_j)A_2(\eta_j)$$
$$\left.+\int_0^1 \mathbf{Y}_4(\nu,\eta')A_2(\eta')\,d\eta'\right] \tag{74b}$$

$$A_2(\eta_i)\left[1-\frac{1}{2}\eta_i N_2^{-1}(\eta_i)\tilde{\mathbf{X}}_2(\eta_i)\mathbf{H}_2^{-1}(\eta_i)\mathbf{J}_5(\eta_i,\eta_i)\mathbf{C}_2\mathbf{U}_2(\eta_i)\right]$$
$$= A^0(\eta_i)+\eta_i N_2^{-1}(\eta_i)\tilde{\mathbf{X}}_2(\eta_i)\left[\sum_{j=1}^{M_2}(1-\delta_{ij})\mathbf{Y}_5(\eta_i,\eta_j)A_2(\eta_j)\right.$$
$$+\int_0^1 \mathbf{Y}_6(\eta_i,\eta')A_2(\eta')\,d\eta'+\sum_{j=1}^{M_1}\mathbf{Y}_7(\eta_i,\nu_j)A_1(\nu_j)$$
$$\left.+\int_0^1 \mathbf{Y}_8(\eta_i,\nu')A_1(\nu')\,d\nu'\right] \tag{75a}$$

and

$$A_2(\eta) = A_2^0(\eta)+\eta N_2^{-1}(\eta)\tilde{\mathbf{X}}_2(\eta)\left[\sum_{j=1}^{M_2}\mathbf{Y}_5(\eta,\eta_j)A_2(\eta_j)\right.$$
$$+\int_0^1 \mathbf{Y}_6(\eta,\eta')A_2(\eta')\,d\eta'$$
$$\left.+\sum_{j=1}^{M_1}\mathbf{Y}_7(\eta,\nu_j)A_1(\nu_j)+\int_0^1 \mathbf{Y}_8(\eta,\nu')A_1(\nu')\,d\nu'\right] \tag{75b}$$

where A^0 and A_2^0 are constant terms due to the source, the $\mathbf{J}$'s are 2×2 matrices of exponential functions and the $\mathbf{Y}$'s are known vectors and matrices involving the H matrices and exponential functions similar to the expressions that appear in Eqs (45) and (49). We list these functions in the Appendix.

V NUMERICAL RESULTS

Computations were performed on an IBM 370/155 computer in double precision arithmetic using standard Gaussian quadrature sets to represent integrals. Our results reported here are obtained using a 20 and a 40 point set in the intervals $(0, 1/\sigma)$ and $(1/\sigma, 1)$ respectively. The accuracy of iterative solutions depends on the quadrature sets used. Because of long computation times we did not use any higher order quadrature sets and the accuracy of our results is generally five or six significant figures as verified by calculating moments of various orders of the equations for the boundary and interface conditions.

V A Cross Section Sets

Several cross section sets for two group calculations are available in the literature[9,10,11]. However since in two media problems the group energies must be compatible we have generated the cross section sets given in Tables I and II using the XSDRN code[19]. The entire energy range $(0 < E < 15$ MeV$)$ is divided at 0.3 eV (0.2994 eV in the code) to give thermal and fast energy groups

Group 1 $\quad E < 0.3$ eV

Group 2 $\quad E \geq 0.3$ eV

This dividing energy may be considered too low for a conventional division of thermal and fast groups. We have selected this value to keep the matrix **G** from becoming triangular since for higher dividing energies the up scattering cross section becomes quite small. Sets 1 through 4 are calculated for infinite media. To calculate

[19] N. M. GREEN and C. N. CRAVEN, XSDRN: A Discrete Ordinate Spectral Averaging Code, ORNL/TM-2500, Oak Ridge National Laboratory (1969).

TABLE I

Definition of the Cross Section Sets

Set	Material
1	H O
2	H O + B B/H = 3/2000
3	H_2O + ^{235}U U/H = 1/1000
4	H_2O + ^{2} U U/H = 1/500
5	^{35}U

set 5 we took the microscopic cross sections for ^{235}U from the calculation of set 3 and multiplied them by the normal density of uranium The fission cross sections are taken to be zero for use in the cell problem

The elements of the matrices Σ and Q are calculated from the data sets as follows

$$\sigma = \sigma_1/\sigma_2 \qquad\qquad q_{ij} = (\sigma_{ij} + \chi_i \bar{\nu}_j \sigma_{fj})/2\sigma_2$$

V B The Two Slab Problem

We consider three cases of incident flux

$$f(\mu) = 2\begin{bmatrix}0\\1\end{bmatrix} \qquad \text{case 1}$$

$$f(\mu) \quad 3\mu\begin{bmatrix}0\\1\end{bmatrix} \qquad \text{case 2}$$

and

$$f(\mu) = 4\mu^2\begin{bmatrix}0\\1\end{bmatrix} \qquad \text{case 3}$$

and use sets 1 and 2 for sample calculations

The scalar fluxes are defined by

$$\begin{bmatrix}\phi_1(x)\\ \phi\ (x)\end{bmatrix} = \int_{-1}^{1} I(x\ \mu)\,d\mu$$

We report in Table III the group 1 scalar flux for cases 1 and 2 and in Fig 1 the scalar fluxes for case 3 All results are for σ_1 σ_2 1 and we use the notation (set i set j) to denote that set i is used for medium 1 and set j for medium 2 The group 2 scalar flux is unchanged to the third digits with the reversal of the media The number of iterations is ~35 and the computation time for one case is ~12 min

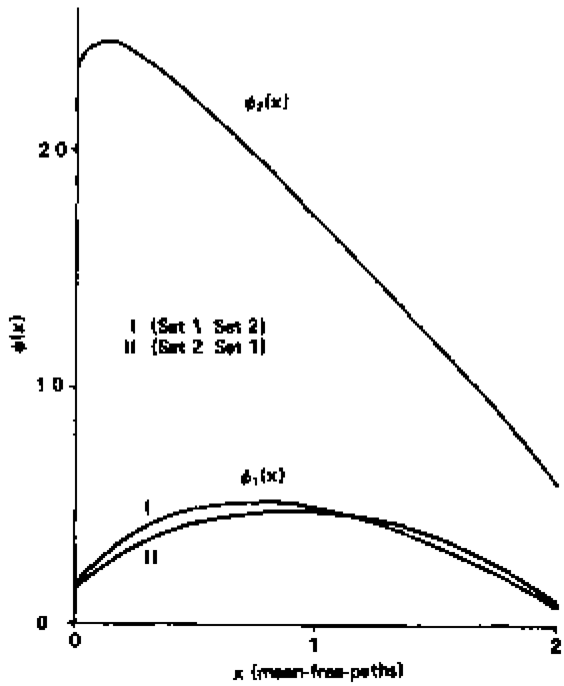

Fig 1 The scalar fluxes in two slabs with an incident flux

$$\Psi(0,\mu) = 4\mu^2\begin{bmatrix}0\\1\end{bmatrix}$$

TABLE II

Macroscopic Cross Sections and the Discrete Eigenvalues

	Set 1	Set 2	Set 3	Set 4	Set 5
σ_1	2 9865	2 9684	2 9727	2 9628	25 826
σ_2	0 86798	0 86731	0 86721	0 86655	1 2763
σ_{11}	2 9678	2 8876	2 9163	2 8751	0 59334
σ_{12}	0 04749	0 04688	0 04635	0 04536	0 00001421
σ_{21}	0 000336	0 00106	0 000767	0 00116	0 000003357
σ_{22}	0 83975	0 83912	0 83882	0 83807	0 41877
$\bar{\nu}_1\sigma_{f1}$	0 0	0 0	0 07391	0 14324	0 0
$\bar{\nu}_2\sigma_{f2}$	0 0	0 0	0 00209	0 00412	0 0
χ_1	0 0	0 0	0 0	0 0	0 0
χ_2	0 0	0 0	1 0	1 0	0 0
Discrete Eigenvalues					
	2 604020 1 122979	2 551909 1 070095	i4 721086 1 152128	13 437681 -	1 004466

TABLE III

The Group 1 Scalar Flux in Two Slabs with an Incident Flux

x	$\phi_1(x)$	$\phi_1(x)^b$	$\phi_1(x)$	$\phi_1(x)^d$
0 0	0 16816	0 14546	0 16286	0 14054
0 2	0 36402	0 31226	0 35078	0 30612
0 4	0 46469	0 39937	0 46118	0 39702
0 6	0 51356	0 44804	0 51434	0 44891
0 8	0 52003	0 46878	0 52444	0 47433
1 0	0 48805	0 47039	0 49492	0 47875
1 2	0 43299	0 44800	0 44108	0 45801
1 4	0 36714	0 39637	0 37541	0 40663
1 6	0 28974	0 32156	0 29716	0 33078
1 8	0 20071	0 22618	0 20633	0 23313
2 0	0 08688	0 09781	0 08939	0 10089

Case 1 (set 1 set 2)
[b]Case 1 (set 2 set 1)
Case 2 (set 1 set 2)
[d]Case 2 (set 2 set 1)

V C The Critical Problem

We consider two cases

Case	Core	Reflector
1	Set 3	Set 1
2	Set 4	Set 1

For case 2 we considered only the case of infinite reflector but for case 1 several reflector thicknesses are considered Our results for the case of infinite reflector are shown in Table IV together with percent errors of P_N approximation results The P approximation gives slightly larger critical sizes but the P_3 approximation is quite good for the cases considered here We report in Table V our results for finite reflectors where γ is the reflector thickness in mean free paths and in Fig 2 the

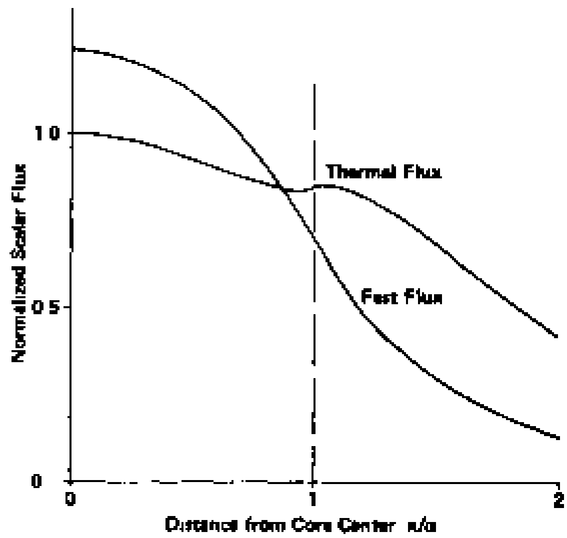

Fig 2 The scalar fluxes in a slab reactor with an infinite reflector

scalar fluxes for case 1 with an infinite reflector The number of iterations is 51 and 39 and the computation time is ~61 and 53 min for cases 1 and 2 respectively with the infinite reflector The long computation time is partly due to the fact that most of the calculation must be performed in complex mode

We have also considered two cases of fast reactor model using the cross section sets for ^{235}U ^{238}U and ^{239}Pu given in Ref 21 The convergence is quite slow and we have not pursued to obtain results of reportable accuracy

V D The Cell Problem

We use set 5 for the fuel and set 1 for the moderator to calculate the thermal disadvantage factor defined as

$$\xi = (a_1/a_2)\int_a^{\gamma} \phi_{21}(x)\,dx \Big/ \int_0^{a_1} \phi_{12}(x)\,dx$$

where ϕ_{i1} is the thermal group scalar flux in medium i In the fuel region we take $\overline{\nu\sigma_f} = 0$ and in the moderator we consider uniform sources of thermal neutrons

$$S = \begin{bmatrix} 1/a_2 \\ 0 \end{bmatrix}$$

Figure 3 shows the thermal flux for three fuel thicknesses with $a_1 = 0$ 2 and we report in Table VI the thermal disadvantage factor for several cell sizes The S_N results were obtained by the ANISN code [22] For the exact calculation the number of iterations and the computation time are of the same order as for the two slab problem For the S_N calculation the computation time is from 12 s (S_4) to 30 s (S_{16}) with 20 spatial mesh points in each of the fuel and moderator regions All S_N results for the disadvantage factor are smaller than our results

TABLE IV

Critical Half Thickness of the Core and Percent Errors of P_N Approximations for the Case of the Infinite Reflector

Case	Exact a	Percent Errors	
		P_1	P_3
1	4 15767	1 0	<0 1
2	2 1028	1 9	<0 1

TABLE V

Critical Half Thickness for Case 1 with a Finite Reflector

Reflector thickness γ	0	1	2	3	5
Core half thickness a	6 85725	5 94147	5 22752	4 75065	4 31486

[21] Reactor Physics Constants ANL-5800 Table 7 5 Argonne National Laboratory (1963)

[22] W W ENGLE, Jr A User s Manual for ANISN A One-Dimensional Discrete Ordinates Transport Code with Anisotropic Scattering," K 1693 Oak Ridge Gaseous Diffusion Plant (1967)

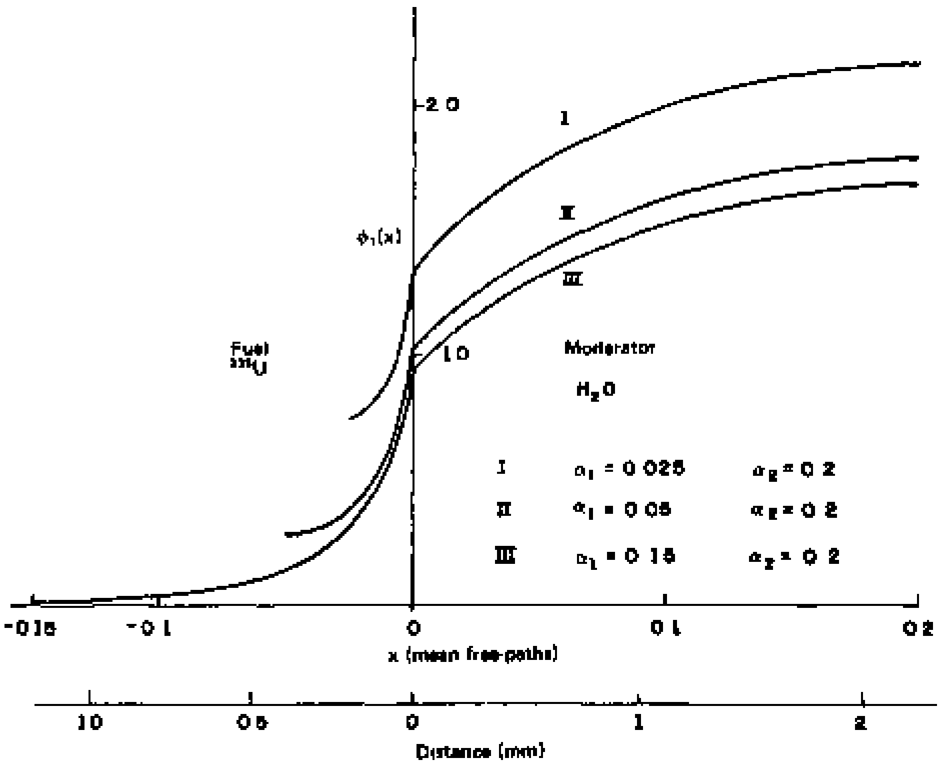

Fig. 3 The thermal flux in U^{235} H_2O cells

For smaller cell sizes the convergence is faster if the equations for the discrete coefficients are derived simply by applying the orthogonality theorem to Eqs. (68) and (69), i.e., without steps 2 and 3 applied to the equations for the continuum coefficients. This seems to be due to the factors that appear in the denominator in Eqs. (70) through (73). The computer program based on Eqs. (74) and (75) can be modified to include this case with an addition of a few statements.

VI. CONCLUSION

For the 15 years or so since its introduction, the singular eigenfunction expansion method has received considerable attention among students of transport theory as an elegant approach to exact analytical solutions. In the past few years, however, interest in this approach has been declining due to several factors such as

1. its limitation to plane geometry
2. availability of highly accurate numerical methods
3. developments of other analytical methods
4. its failure to solve even the highly idealized model problems in two media and two groups due to the ever present singularities.

This Note has two purposes. First, we have shown that the singularity problem can be eliminated by an extension of the established basic method, i.e., using only the half range orthogonality theorem. Clearly the analytical and computational tasks are quite involved and this solution method is not suitable for routine calculations. However, it can be of use in support of benchmark activities in providing numerical solutions to well defined model problems. We have therefore as the second aim of this Note defined three problems and reported their solutions based on an exact theory.

TABLE VI

Thermal Disadvantage Factor for Two-Slab Cells and Percent Errors of S_N Results

α_1	α_2	Exact ξ	Percent Errors			
			S_2	S_4	S_8	S_{16}
0.25	0.5	20.079	15.8	2.9	0.86	0.33
0.15	0.5	12.055	15.6	2.9	0.86	0.32
0.05	0.5	4.2910	18.6	3.1	0.91	0.36
0.15	0.2	9.7457	26.3	6.0	3.2	0.76
0.05	0.2	3.4757	27.9	8.4	3.3	0.78
0.025	0.2	2.1489	27.0	10.4	3.8	0.84

APPENDIX

The Y functionals that appear in Eqs. (74) and (75) are as follows

$$Y_1(\xi, \nu_j) = \frac{\nu_j}{\xi + \nu_j} H^{-1}(\nu_j) J_1(\xi, \nu_j) C^{-1} U(\nu_j)$$
$$- \frac{\nu_j}{\nu_j - \xi} H_1^{-1}(-\nu_j) J_2(\xi, \nu_j) C^{-1} U_1(\nu_j)$$
$$Y(\xi, \nu) = \frac{\nu}{\xi + \nu} H_1^{-1}(\nu) J_1(\xi, \nu) C^{-1} - \frac{\nu}{\nu - \xi} C_1 \widetilde{H}_1(\nu)$$
$$\times [\lambda(\nu) C_1^{-1} J(\xi, \nu) C_1 - J(\xi, \nu) \Phi_1(\nu) \lambda_1(\nu)]$$
$$Y_3(\xi, \eta_j) = \left[\frac{\sigma_1 \eta_j}{\sigma_2 \eta_j + \sigma_1 \xi} H^{-1}(\sigma_2 \eta_j / \sigma_1) J_3(\xi, \eta_j) k_1 + \frac{\eta_j}{\eta_j + \xi} H_1^{-1}(\eta_j) J_3(\xi, \eta_j) k_2\right] G C_2 U_2(\eta_j)$$
$$\left[\frac{\sigma_1 \eta_j}{\sigma_2 \eta_j - \sigma_1 \xi} H_1^{-1}(-\sigma_2 \eta_j / \sigma_1) J_4(\xi, \eta_j) k_1 + \frac{\eta_j}{\eta_j - \xi} H_1^{-1}(-\eta_j) J_4(\xi, \eta_j) k_2\right] G C_2 U_2(\eta_j)$$
$$Y_4(\xi, \eta) = \left[\frac{\sigma_1 \eta}{\sigma_2 \eta + \sigma_1 \xi} H^{-1}(\sigma_2 \eta / \sigma_1) J_3(\xi, \eta) k_1 + \frac{\eta}{\eta + \xi} H_1^{-1}(\eta) J_3(\xi, \eta) k_2\right] G C_2$$
$$C_1 \left[\frac{\sigma_1 \eta}{\sigma_2 \eta - \sigma_1 \xi} \widetilde{H}_1(\sigma_2 \eta / \sigma_1) \lambda(\sigma_2 \eta / \sigma_1) C_1^{-1} J_4(\xi, \eta) k_1 + \frac{\eta}{\eta - \xi} \widetilde{H}(\eta) \lambda_1(\eta) C_1^{-1} J_4(\xi, \eta) k_2\right] G C_2$$
$$+ C_1 \left[\frac{\sigma_1 \eta}{\sigma_2 \eta - \sigma_1 \xi} \widetilde{H}_1(\sigma_2 \eta / \sigma_1) J_4(\xi, \eta) k_1 + \frac{\eta}{\eta - \xi} \widetilde{H}_1(\eta) J_4(\xi, \eta) k_2\right] \Phi_2(\eta) G \lambda_2(\eta)$$
$$Y(\xi, \eta_j) = \frac{\eta_j}{\eta_j + \xi} H^{-1}(\eta_j) J_5(\xi, \eta_j) C_2 U(\eta_j)$$
$$\frac{\eta_j}{\eta_j - \xi} H_2^{-1}(-\eta_j) J_6(\xi, \eta_j) C_2 U_2(\eta_j)$$
$$Y_6(\xi, \eta) = \frac{\eta}{\eta + \xi} H_2^{-1}(\eta) J_5(\xi, \eta) C_2 - \frac{\eta}{\eta - \xi} C_2 \widetilde{H}_2(\eta)$$
$$\times [\lambda_2(\eta) C_2^{-1} J_6(\xi, \eta) C_2 - J_6(\xi, \eta) \Phi_2(\eta) \lambda_2(\eta)]$$
$$Y_7(\xi, \nu_j) = \left[\frac{\sigma_2 \nu_j}{\sigma_1 \nu_j + \sigma_2 \xi} H_2^{-1}(\sigma_1 \nu_j / \sigma_2) J_7(\xi, \nu_j) k + \frac{\nu_j}{\nu_j + \xi} H_2^{-1}(\nu_j) J_7(\xi, \nu_j) k_2\right] G^{-1} C_1 U_1(\nu_j)$$
$$\left[\frac{\sigma_2 \nu_j}{\sigma_1 \nu_j - \sigma_2 \xi} H_2^{-1}(-\sigma_1 \nu_j / \sigma_2) J_8(\xi, \nu_j) k_1 + \frac{\nu_j}{\nu_j - \xi} H_2^{-1}(-\nu_j) J_8(\xi, \nu_j) k_2\right] G^{-1} C_1 U(\nu_j)$$
$$Y(\xi, \nu) = \left[\frac{\sigma_2 \nu}{\sigma_1 \nu + \sigma_2 \xi} H_2^{-1}(\sigma_1 \nu / \sigma_2) J_7(\xi, \nu) k_1 + \frac{\nu}{\nu + \xi} H_2^{-1}(\nu) J_7(\xi, \nu) k_2\right] G^{-1} C_1$$
$$C_2 \left[\frac{\sigma_2 \nu}{\sigma_1 \nu - \sigma_2 \xi} \widetilde{H}_2(\sigma_1 \nu / \sigma_2) \lambda_2(\sigma_1 \nu / \sigma_2) C_2^{-1} J_8(\xi, \nu) k_1 + \frac{\nu}{\nu - \xi} \widetilde{H}_2(\nu) \lambda_2(\nu) C_2^{-1} J_8(\xi, \nu) k_2\right] G^{-1} C_1$$
$$+ C_2 \left[\frac{\sigma_2 \nu}{\sigma_1 \nu - \sigma_2 \xi} \widetilde{H}_2(\sigma_1 \nu / \sigma_2) J_8(\xi, \nu) k_1 + \frac{\nu}{\nu - \xi} \widetilde{H}_2(\nu) J_8(\xi, \nu) k_2\right] \Phi_1(\nu) G^{-1} \lambda(\nu)$$

where

$$J_1(x, y) = \begin{bmatrix} E_2(\sigma_1 x/\sigma_2) & E_1(y) & 0 \\ 0 & E_2(x) & E_1(y) \end{bmatrix} J^1(x)$$
$$J_2(x, y) = \begin{bmatrix} E_2(\sigma_1 x/\sigma_2) & 0 \\ 0 & E_2(x) \end{bmatrix} [E_1(x) \quad E_1(y)] J^1(x)$$
$$J_3(x, y) = \begin{bmatrix} 1 & E_2(\sigma_1 x/\sigma_2) E_2(y) & 0 \\ 0 & 1 & E_2(x) E_2(y) \end{bmatrix} J^1(x)$$
$$J_4(x, y) = \begin{bmatrix} E_2(\sigma_1 x/\sigma_2) & E_2(y) & 0 \\ 0 & E_2(x) - E_2(y) \end{bmatrix} J^1(x)$$
$$J_5(x, y) = \begin{bmatrix} E_1(\sigma_2 x/\sigma_1) & E_2(y) & 0 \\ 0 & E_1(x) & E_2(y) \end{bmatrix} J^2(x)$$
$$J_6(x, y) = \begin{bmatrix} E_1(\sigma_2 x/\sigma_1) & 0 \\ 0 & E_1(x) \end{bmatrix} [E_2(x) \quad E_2(y)] J^2(x)$$
$$J_7(x, y) = \begin{bmatrix} 1 & E_1(\sigma_2 x/\sigma_1) E_1(y) & 0 \\ 0 & 1 & E_1(x) E_1(y) \end{bmatrix} J^2(x)$$
$$J_8(x, y) = \begin{bmatrix} E_1(\sigma_2 x/\sigma_1) & E_1(y) & 0 \\ 0 & E_1(x) & E_1(y) \end{bmatrix} J^2(x)$$

with

$$J^1(x) = \begin{bmatrix} [1 - E_1(x) E_2(\sigma_1 x/\sigma_2)]^{-1} & 0 \\ 0 & [1 - E_1(x) E_2(x)]^{-1} \end{bmatrix}$$
$$J^2(x) = \begin{bmatrix} [1 - E_1(\sigma_2 x/\sigma_1) E_2(x)]^{-1} & 0 \\ 0 & [1 - E_1(x) E_2(x)]^{-1} \end{bmatrix}$$